Free resolutions induced by Pommaret-like bases over Clements–Lindström rings

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Setting

We work with a polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$ and homogeneous polynomial ideals in this ring; we assume a fixed term ordering has been chosen and thus we can determine the leading term $\mathrm{lt}(f)$ of any nonzero polynomial f. We write $\mathcal{T}$ for the set of all terms $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n} \in \mathcal{R}$.

Involutive divisions and bases

Involutive division L: Restricted division relation for terms with combinatorial properties [4].

- Given set $H \subset \mathcal{T}$ and $\mathbf{x}^\mu \in H$, assigns multiplicative variables $L(\mathbf{x}^\mu, H) \subseteq \{x_1, \dots, x_n\}$.
- The involutive cones $\mathcal{C}_L(\mathbf{x}^\mu, H) = \{\mathbf{x}^\nu \mid \mathbf{x}^\nu/\mathbf{x}^\mu \in \mathbb{K}[L(\mathbf{x}^\mu, H)]\}$ intersect trivially for $\mathbf{x}^\mu \in H$.
- Filter axiom: If $H \supseteq H'$, then $L(\mathbf{x}^\mu, H') \supseteq L(\mathbf{x}^\mu, H)$.
- Polynomial set $H \subset \mathcal{I} \trianglelefteq \mathcal{R}$ *L-involutive basis* of ideal $\mathcal{I}$, if

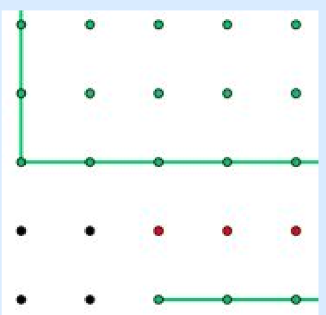
$$\bigsqcup_{\mathbf{x}^\mu \in \mathrm{lt}(H)} \mathcal{C}_L(\mathbf{x}^\mu, \mathrm{lt}(H)) = \mathrm{lt}(\mathcal{I}).$$

Every involutive basis is a Gröbner basis.

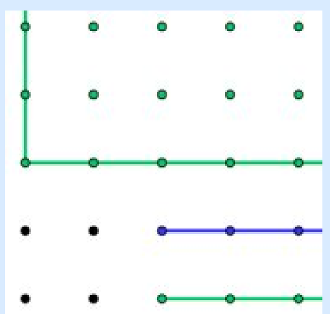
Pommaret bases and quasi-stable ideals

Pommaret division P: term $\mathbf{x}^\mu$ with $\mu = (0, \dots, 0, \mu_k, \dots, \mu_n)$ and $\mu_k > 0$ $\rightsquigarrow P(\mathbf{x}^\mu) = \{x_1, \dots, x_k\}$. Define $\mathrm{cls}(\mathbf{x}^\mu) = \min\{k \mid \mu_k > 0\}$.
Not every monomial ideal $\mathcal{I}$ has a finite Pommaret basis. A monomial ideal with finite Pommaret basis is called quasi-stable. A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if its leading ideal with respect to a term-order is quasi-stable.

- For example, consider $\mathcal{I} = \langle x^2, y^2 \rangle \trianglelefteq \mathbb{K}[x, y]$, we have $\mathrm{cls}(x^2) = 1$, $\mathrm{cls}(y^2) = 2$.
- We obtain a Pommaret basis by adding the generator x^2y.



$\mathcal{I}$ and the Pommaret cones of x^2 and y^2.



The Pommaret cone of x^2y added.

Involutive-like divisions

The *Janet-like division* was discovered by Gerdt and Blinkov [5].

- Let $U = \{t_1, \dots, t_m\} \subset \mathcal{T}$.
- For $k \in \{1, \dots, n\}$ and $d_k, \dots, d_n \geq 0$, write $U_{[d_k, \dots, d_n]} := \{u \in U \mid \deg_k(u) = d_k, \dots, \deg_n(u) = d_n\}$. Also, $U_{[]} := U$.
- For $u \in U$, a pure power $x_i^{k_i}$ with

$$k_i = \min\left\{\deg_i(v) - \deg_i(u) \mid v, u \in U_{[d_{i+1}, \dots, d_n]}, \deg_i(v) > \deg_i(u)\right\}$$

  is called a non-multiplicative power of u.
- The set $\mathrm{NMP}(u, U)$ collects all non-multiplicative powers of $u \in U$.
- J-multipliers for $u \in U$: terms not divisible by any $x_i^{k_i} \in \mathrm{NMP}(u, U)$.
- $u \in U$ is Janet-like divisor of $w \in \mathcal{T}$, if w/u is a J-multiplier for u.

The Pommaret-like division assigns $\mathrm{NMP}_P(u, U)$ as follows:

- All Janet-like non-multiplicative powers $x_a^{p_a}$ with $a > \mathrm{cls}(u)$.
- The Janet-like multiplier variables x_b with $b > \mathrm{cls}(u)$.

Relative Gröbner and Pommaret-like bases

We work in the quotient ring $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is a homogeneous ideal with minimal Gröbner basis G.

- Ideal $\hat{\mathcal{J}} \trianglelefteq \mathcal{R}/\mathcal{I} \rightsquigarrow \hat{\mathcal{J}} = \mathcal{J}/\mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$.
- $H = \{h_1, \dots, h_k\} \subset \mathcal{J}$ Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ if

$$\langle \mathrm{lt}(H) \rangle + \mathrm{lt}(\mathcal{I}) = \mathrm{lt}(\mathcal{J}).$$

- H Gröbner basis of $\mathcal{J} \implies \{\mathrm{NF}_G(h) \mid h \in H \setminus \mathcal{I}\}$ Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.

We derive a Pommaret-like division $P_\mathcal{I}$ in monomial quotient rings $\mathcal{R}/\mathcal{I}$.

- Relative Pommaret-like cones $\mathcal{C}_{P,\mathcal{I}}(\mathbf{x}^\mu, H)$.
- $H = \{h_1, \dots, h_k\} \subset \mathcal{J} \setminus \mathcal{I}$ Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$, if $\bigsqcup_{h \in H} \mathcal{C}_{P,\mathcal{I}}(\mathrm{lt}(h), \mathrm{lt}(H)) = \mathrm{lt}(\mathcal{J}) \setminus \mathrm{lt}(\mathcal{I})$.
- Exists for $\mathcal{I}$ in relative quasi-stable position.
- This basis is in general smaller than the (relative) Pommaret basis of the same ideal.

Free Resolutions

Free resolution $\mathbf{F}$ of homogeneous $\mathcal{J} \trianglelefteq \mathcal{R}/\mathcal{I}$:
Finitely generated free $\mathcal{R}/\mathcal{I}$-modules $F_0, F_1, \dots$ and homogeneous $\mathcal{R}/\mathcal{I}$-linear maps $\delta_0, \delta_1, \delta_2, \dots$ as in the following diagram:

$$\mathbf{F}: \quad \cdots \xrightarrow{\delta_{m+2}} F_{m+1} \xrightarrow{\delta_{m+1}} F_m \xrightarrow{\delta_m} F_{m-1} \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} \mathcal{J} \to 0,$$

with $\mathrm{im}(\delta_0) = \mathcal{J}$ and $\mathrm{im}(\delta_{m+1}) = \ker(\delta_m)$ for all $m \geq 0$.

- *Differential* of $\mathbf{F}$: Collection $\{\delta_0, \delta_1, \delta_2, \dots\}$.
- δ_m is represented by matrix D_m with entries in $\mathcal{R}/\mathcal{I}$, and $D_m \cdot D_{m+1} = 0$.
- Resolution $\mathbf{F}$ is *minimal*: $\iff$ no constant terms $\neq 0$ in matrices D_m.
- Minimal free resolution of $\mathcal{I}$ contains invariants of $\mathcal{I}$ in the form of *Betti numbers*. (see e.g. [7])
- The Betti numbers are bigraded. The Poincaré series $\mathrm{PS}_{\mathcal{J}/\mathcal{I}}(u, s)$ encodes the Betti numbers; u stands for term degree and s for homological degree.

Free resolutions from relative Pommaret-like bases

Monomial ideal $\mathcal{I} \trianglelefteq \mathcal{R}$ is *Clements-Lindström (C-L)*, if it is generated by $G = \{x_i^{a_i}, x_{i+1}^{a_{i+1}}, \dots, x_n^{a_n}\}$ with $2 \leq a_n \leq a_{n-1} \leq \cdots \leq a_{i+1} \leq a_i$. We call $\mathcal{R}/\mathcal{I}$ a *Clements-Lindström ring*.
Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{R}$, H minimal Pommaret-like basis of $\mathcal{J}$ relative to $\mathcal{I}$. For each $h \in H$ take:

1. Syzygies $\mathbf{S}_{h,x_i}$ ($x_i^{p_i} \in \mathrm{NMP}_{P_\mathcal{I}}(\mathrm{lt}(h), \mathrm{lt}(H))$) induced by non-multiplicative powers,
2. Syzygies $\mathbf{S}_{h,\mathbf{x}^\mu}$, where $\mathbf{x}^\mu$ is a $P_\mathcal{I}$-multiplier, induced by annihilating $\mathrm{lt}(h)$ mod $\mathrm{lt}(\mathcal{I})$.

Choose suitable module term ordering on $(\mathcal{R}/\mathcal{I})^{|H|}$. Then
$H_1 = \{\mathbf{S}_{h,x_i} \mid h \in H\} \cup \{\mathbf{S}_{h,\mathbf{x}^\mu} \mid h \in H\}$ minimal Pommaret-like basis of $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$.
Iteration $\rightsquigarrow \mathcal{R}/\mathcal{I}$-free resolution (not minimal in general)

$$\cdots \to F_1 = (\mathcal{R}/\mathcal{I})^{|H_1|} \to F_0 = (\mathcal{R}/\mathcal{I})^{|H|} \to \mathcal{J}/\mathcal{I} \to 0.$$

Minimal free resolutions from relative Pommaret-like bases

$\mathcal{I} \trianglelefteq \mathcal{R}$ C-L ideal, $\mathcal{J} \supset \mathcal{I}$ monomial ideal generated by minimal relative Pommaret-like basis $H \subset (\mathcal{J} \setminus \mathcal{I}) \cap \mathcal{T}$. If H is the minimal generating set of $\mathcal{J}$ relative to $\mathcal{I}$ and for each $t \in H$ and $x_a^{p_a} \in \mathrm{NMP}_{P_\mathcal{I}}(t, H)$, the unique $P_\mathcal{I}$-divisor $s \in H$ of $t \cdot x_a^{p_a}$ fulfils $\mathrm{cls}(s) > \mathrm{cls}(t)$, then the free resolution of $\mathcal{J}/\mathcal{I}$ over $\mathcal{R}/\mathcal{I}$ induced by H is minimal.

Module bases and Poincaré series

Given: Pommaret-like basis H relative to $\mathcal{I} = \langle x_i^{a_i}, x_{i+1}^{a_{i+1}} \dots, x_n^{a_n} \rangle$. Write $\mathcal{S} := \{x_i, x_{i+1}, \dots, x_n\}$.

- The free module F_0 has basis $\{\mathbf{e}_\alpha \mid h_\alpha \in H\}$. Write $t_\alpha = \mathrm{lt}(h_\alpha)$ for each $h_\alpha \in H$.
- Write $\mathcal{S}_\alpha$ for the set of variables appearing in the generators of $\mathcal{J}_\alpha = \langle x_k^{d_k} \mid x_k^{d_k} \cdot \mathbf{e}_\alpha \in \mathrm{lt}(\mathrm{Syz}(H)) \rangle$.
- The free module F_r ($r \geq 1$), has basis elements $\mathbf{e}_{\alpha,\mathbf{x}^\mu}$, where $\mathbf{x}^\mu$ is a term of degree r with $\mathrm{cls}(\mathbf{x}^\mu) \geq \mathrm{cls}(t_\alpha)$. Moreover, $\mathbf{x}^\mu$ is supported on $\mathcal{S}_\alpha$, and its projection onto $\mathcal{S}_\alpha \setminus \mathcal{S}$ is squarefree.

For the Betti numbers and the Poincaré series, we obtain

$$\mathrm{rank}(F_r) = \sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} \neq \emptyset}} \sum_{j=0}^{\min\{r, |\mathcal{S}_\alpha \setminus \mathcal{S}|\}} \binom{|\mathcal{S}_\alpha \setminus \mathcal{S}|}{j} \cdot \binom{|\mathcal{S}_\alpha \cap \mathcal{S}| + r - j - 1}{|\mathcal{S}_\alpha \cap \mathcal{S}| - 1} + \sum_{\substack{h_\alpha \in H \\ \mathcal{S}_\alpha \cap \mathcal{S} = \emptyset}} [r \leq |\mathcal{S}_\alpha|] \binom{|\mathcal{S}_\alpha|}{r},$$

$$\mathrm{PS}_{\mathcal{J}/\mathcal{I}}(u, s) = \sum_{h_\alpha \in H} \left(u^{\deg(t_\alpha)} \cdot \left(1 + \sum_{B \subseteq \mathcal{S}_\alpha} \binom{|\mathcal{S}_\alpha|}{|B|} s^{|B|} \prod_{x_b \in B} u^{d_b} \prod_{x_j \in \mathcal{S}_\alpha \cap \mathcal{S}} \frac{1}{1 - s^2 u^{a_j}} \right) \right).$$

An example

If $\mathcal{I} = \langle y^4, z^5 \rangle$ and $\mathcal{J} = \langle \mathcal{I}, x^2y^3, xy^2z^2, y^3z^2, z^3 \rangle$, then $H = \{x^2y^3, xy^2z^2, y^3z^2, z^3\}$ is the minimal relative generating set and a relative Pommaret-like basis of $\mathcal{J}$. $\rightsquigarrow$ minimal free resolution of $\mathcal{J}/\mathcal{I}$ over $\mathcal{P}/\mathcal{I}$, with the first three differential matrices:

$$D_0 = \begin{pmatrix} x^2y^3 & xy^2z^2 & y^3z^2 & z^3 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} y & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & z & 0 & 0 & 0 \\ 0 & -x^2 & -x & 0 & y & z & 0 \\ 0 & 0 & 0 & -xy^2 & 0 & -y^3 & z^2 \end{pmatrix}, \; D_2 = \begin{pmatrix} y^3 & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y & z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^3 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & z^4 & 0 & 0 & 0 & 0 \\ 0 & -x^2 & 0 & xy^2 & 0 & 0 & y^3 & z & 0 & 0 \\ 0 & 0 & x^2z^2 & 0 & x & 0 & 0 & -y & z^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy^2z^2 & 0 & 0 & y^3z^2 & z^3 \end{pmatrix}.$$

For the Betti numbers, we obtain

$$\mathrm{rank}(F_r) = 3\binom{1+r}{1} + \binom{r}{0} = 4 + 3r.$$

Connection to other constructions

The construction contains as special cases:

- The Eliahou–Kervaire [2] resolution of stable monomial ideals in $\mathcal{R}$,
- The Pommaret–Seiler [9] resolution of quasi-stable ideals in $\mathcal{R}$,
- The Gasharov–Murai–Peeva [3] resolution of squarefree Borel ideals in zero-dimensional C-L rings,
- After a suitable variable permutation, Tate's resolution of irreducible monomial ideals in C-L rings.

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